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Simple Kinetic Model of the Cathode Fall in a Low Pressure He Discharge†

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Abstract

A simple model of a DC discharge is considered. Simplifying assumptions allow the kinetic equations to be solved analytically. The resulting electric field, ionization and excitation rates are compared with experiments and numerical simulations of DC discharges.

1. Introduction

Although the cathode fall (CF) regions is the most fundamental region of a DC discharge there is still no theory that describes it. Hydrodynamic theories use the first Townsend coefficient, which is a function of E/N, where E is the electric field strength and N is the gas density, and they fail to explain the distribution of excitation and ionization rates that have maxima at the end of the cathode fall. There are some numerical simulations of a discharge [1, 2], however a simple physical model is desirable. We present a simple kinetic model which, despite its simplicity, can describe several important features of a DC discharge. We limit our model to the case of a very high electric field at the cathode (or low gas pressure) which was recently very carefully investigated experimentally and numerically [1, 3, 4]. The other case of a moderate electric field will be described in a future publication.

We have to solve the system of kinetic equations for ions and electrons. These equations are coupled through the Poisson equation for electric field, and the fluid equations for neutral species (ground state atoms, metastables, excited atoms). We want to find the distribution functions of charged particles and the resulting electric field. In most cases the number of neutral atoms, which are in the ground state, greatly exceeds their number in metastable and excited states as well as the number of charge particles. Thus, one can neglect the electron (ion) collisions with all species except with atoms in the ground state. This means that we can solve the Boltzmann equations for ions and electrons independently and they only couple with the electric field through the Poisson equation. Another simplification is that in most of the cathode fall region the ion density greatly exceeds the electron density (since $n_i \sim j_i/u_i \gg n_e \sim j_e/u_e$, where $n_{e,i}$, $j_{e,i}$, $u_{e,i}$ are electron and ion densities, current densities and drift velocities, respectively). Thus, we can neglect the electron density in the Poisson equation. With these approximations we can solve the Boltzmann equation for ions and use the resulting distribution function to solve the Poisson equation for the electric field. We can then solve the electron Boltzmann equation using this electric field.

2. Electric field

Consider a low pressure discharge ($E/N > 1000\,\mathrm{Td}$ for He, $N\sigma_{tot}d > 1$) with a noble gas between two parallel electrodes. Choose x to be the coordinate along the electric field, where $x = 0$ defines the cathode and x is negative between the electrodes. One can easily find the ion distribution function by realizing that the most important collisional process for ions is charge exchange with neutral atoms. In this case the ion motion is nearly one dimensional. Typically, the electric field varies on a scale length much larger than the mean free path for the charge exchange process, and the ion distribution function is similar to that given by Den Hartog *et al.* [1]:

$$f_i(v_x, v_y, v_z) = C(x)\eta(v_x)\exp\left(-\frac{Mv_x^2 N\sigma_{cx}}{2eE}\right)\delta(v_y)\delta(v_z). \tag{1}$$

In the above, v_x, v_y, v_z are the components of the ion velocity, M and e its mass and charge, N-the gas density and σ_{cx} is the charge exchange cross-section. $\eta(v)$ is the step function: $\eta(v_x) = 0$ if $v_x \leqslant 0$, and $\eta(v_x) = 1$ if $v_x > 0$. We can now find the ion drift velocity $u_i = \langle v_{ix}\rangle$, calculated with the distribution function (1):

$$u_i = \sqrt{\frac{2}{\pi}\frac{eE}{NM\sigma_{cx}}}. \tag{2}$$

Since $n_i = j_i/u_i$, this may be substituted into the Poisson equation. Neglecting n_e compared to n_i in the cathode region we obtain [5]:

$$\frac{\partial E}{\partial x} = 4\pi\frac{j}{1+\gamma}\sqrt{\frac{\pi NM\sigma_{cx}}{2eE}}, \tag{3}$$

where j is the total current and γ is the secondary emission coefficient: $j_i = j/(1+\gamma)$. We define $\phi(x)$ to be the electric potential and integrate this equation with boundary conditions $\phi(0) = -U$ and $\phi(x) = 0$ at the point $x = -d$ where $E = 0$. The solution is

$$E(x) = (\tfrac{3}{2}C)^{2/3}(d+x)^{2/3}, \quad \phi(x) = -\tfrac{3}{5}(\tfrac{3}{2}C)^{2/3}(d+x)^{5/3}, \quad \text{if } d+x>0 \tag{4a}$$

and

$$E(x) = 0, \quad \phi(x) = 0, \quad \text{if } d+x \leqslant 0. \tag{4b}$$

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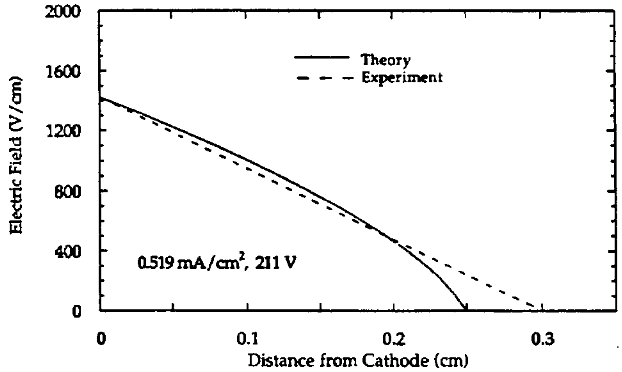

Fig. 1. A comparison of measured and calculated electric fields in the cathode fall region of the discharge.

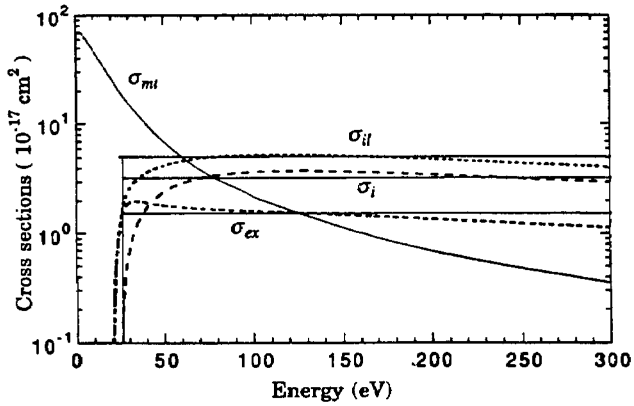

Fig. 2. Momentum transfer, ionization, excitation and total inelastic integral cross-sections used in this work. Continuous lines are the approximations used for the actual cross-sections (dotted lines).

In the above, d is the distance from the cathode to the point where $E = 0$, and

$$C = 2\pi \frac{j}{1+\gamma} \sqrt{\frac{2\pi N M \sigma_{cx}}{e}}. \tag{5}$$

Expressions for $E(x)$ and $\phi(x)$ yield simple relations which are easy to compare with the experiment [1]:

$$d = \frac{5}{3}\frac{U}{E(0)}, \quad E(0) = (\tfrac{5}{2}CU)^{2/5}. \tag{6}$$

It is also useful to define a characteristic length l which can be used in a linear approximation of the electric field:

$$l = \frac{E(0)}{\partial E/\partial x|_{x=0}} = E^{3/2}(0)/C = (\tfrac{5}{2}CU)^{3/5}/C. \tag{7}$$

The theoretical curves $E(x)$ given by eqs (4) and (7) for the parameters given in the Den Hartog *et al.* experiment [1] are compared with their experimentally measured values in Fig. 1. It is seen that agreement is very good throughout most of the CF and differs only in locating the end point of the cathode fall region.

3. Electron distribution function

Now that we have an expression for the electric field we can find the electron distribution function. First, we notice that the electric field used in the experiments [1] was so high that almost all electrons created in the CF region did not attain their equilibrium drift velocity (this feature was most prominent from the other experiments). We can write for the x-component v_x of the electron velocity v:

$$m_e \frac{dv_x}{dt} = eE - mN\sigma_{mt} v v_x - N\sigma_{il} I \frac{v_x}{v}. \tag{8}$$

Note that here we have chosen the positive direction of the x-axis pointing toward the anode so that $E_x = -E(x)$ ($x = 0$ again in the cathode). Here σ_{mt} and σ_{il} are momentum transfer, and total inelastic cross-sections, and I is the ionization potential. In noble gases the difference between the ionization and first excitation thresholds is small, and within the spirit of our approximation we will consider them both to be equal to I. An additional simplification is to treat these cross-section as constants at the energies above the threshold (see Fig. 2). We further assume that during an inelastic collision the primary electron energy loss is equal to the ionization energy (I). Thus, the newly produced electrons appear with zero kinetic energy. If the right side of the equation (8) is always positive, then electrons undergo only positive acceleration and do not reach stationary speed. This happens to all particles with the energy ε if

$$\frac{eE(x)}{N} > 2\sigma_{mt}(\varepsilon)\varepsilon + \sigma_{il}(\varepsilon)I, \tag{9}$$

where $\varepsilon = \varepsilon(x)$ is the electron kinetic energy.

In order to see whether or not this inequality may be satisfied along the trajectory, we can substitute into eq. (9) the potential energy of electron $V(x) = -e\phi(x)$ and use $\varepsilon = V^* - V$, where ϕ^* is the potential at the point where the electron was produced. For the simple linear approximation ($E = E_0\,\xi/l$, where $\xi = l - x$ is the distance between the particle and zero-field point) we obtain the following condition for the electric field,

$$E = E_0\sqrt{V/eU} = \frac{2}{l}\sqrt{VU/e}. \tag{10}$$

In Fig. 3 we plot the left and the right sides of the inequality (9) as a function of the particle kinetic energy (ε). Different curves of the left side correspond to the different points where an electron was produced. It is seen that for the parameters of most of the experiments provided by Lawler, Den Hartog and coworkers, inequality (9) is satisfied. This means that electrons produced far enough from the point where $E = 0$ must form a beam, and we can expect that at

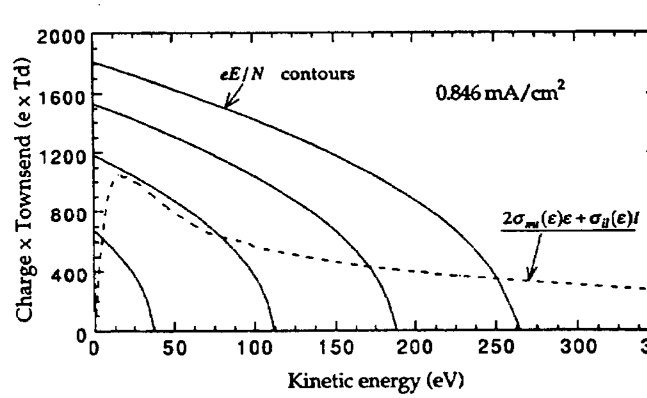

Fig. 3. Parametric plot of eE/N contours and $2\sigma_{mt}(\varepsilon)\varepsilon + \sigma_{il}(\varepsilon)I$ as a function of the electron kinetic energy.

high E/N the electron distribution function is broad in energy and narrow in the angle variable.

Prior to the derivation of the distribution function we point out some distinguishing features of electrons close to the maximum energy. We recall that elastic collisions do not change the particle energy. However, inelastic collisions decrease the energy of the electron by the ionization potential – about 25 eV. From this, we may conclude that there will be no electrons in the energy range between ε_{max} and $\varepsilon_{max} - I$. Thus, if we had a peak at $\varepsilon = \varepsilon_{max}$, which consist of the particles produced at the cathode, then the next peak is at the energy $\varepsilon_{max} - I$. Similarly, the particles from the second peak cannot be in the interval $(\varepsilon_{max} - I, \varepsilon_{max} - 2I)$. This interval is totally determined by the particles from the first peak where $\varepsilon = \varepsilon_{max}$. Only at energies lower than $\varepsilon_{max} - 2I$ (which is about 50 eV lower than ε_{max}), the distribution function is formed by more than one energy shell. The amplitude of the peak at $\varepsilon = \varepsilon_{max}$ depends on the coordinate x in a manner given by $\exp(-\int N\sigma_{il} v \, dx/u_e)$, where u_e is the drift velocity of electrons forming this group.

Before solving the kinetic equation for electrons we point out that at low energies the elastic cross-section is isotropic, and does not depend on the scattering angle. However at higher energies, exceeding 20–30 eV, strong forward scattering is prominent [2, 6, 7]. Thus, we can expand the distribution function of high energy particles in the elastic collision term near $\theta = 0$. After simple calculations we find that instead of the usual elastic collision term $St_{el} = -\sigma_{mt} v(f_e - \langle f_e \rangle)$ we have

$$St_{el} = \frac{Nv\sigma_{tr}}{2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial f_e}{\partial\theta}. \qquad (11)$$

Using this collision term we can write the Boltzmann equation for energetic electrons as follows:

$$\cos\theta \frac{\partial f_e \varepsilon}{\partial x} = \frac{N}{2\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \left(\frac{eE}{N} \sin\theta f_e + \varepsilon\sigma_{mt}(\varepsilon) \frac{\partial f_e}{\partial\theta} \right) + St_{il}(f_e), \qquad (12)$$

where the distribution function f_e and $\varepsilon = mv^2/2$ are considered as functions of coordinates θ, x and the total energy $\mathscr{E} = \varepsilon + V$. In the above, St_{il} is an inelastic collision term:

$$St_{il}(f_e) = N\left(-\sigma_{il}(\varepsilon) v(\varepsilon) f_e(\varepsilon) + \frac{1}{2} \int \sigma_{il}(\varepsilon') f_e(\varepsilon', \theta) v'^3 \, dv' \sin\theta \, d\theta \right) + \delta(\varepsilon) N \int_{\sqrt{(2I/m)}}^{\infty} \sigma_i(v) v f_e(v) \, dv, \qquad (13)$$

$$\varepsilon' \equiv \varepsilon + I, \quad v' \equiv v(\varepsilon').$$

The largest term in equation (12) is the one in the right side of it, which contains differentiation over θ. Equating this term to zero gives the angular distribution of the electron distribution function in the lowest order approximation. We find,

$$f_e \propto \exp\left(\frac{2eE\cos\theta}{N\sigma_{mt}\varepsilon} \right) \propto \exp\left(-\frac{eE}{N\sigma_{mt}\varepsilon} \theta^2 \right). \qquad (14)$$

Since $N\sigma_{mt}\varepsilon/eE \ll 1$ the electron distribution function is concentrated in a cone with the width about $(\Delta\theta)^2 \approx N\sigma_{mt}\varepsilon/eE \ll 1$ near $\theta = 0$. Note, that at high energies* (more than 50 eV) $\sigma_{mt}(\varepsilon) \propto \varepsilon^{-2}$. This means that $(\Delta\theta)^2 \propto \varepsilon^{-1}$ or $(\Delta v_\perp)^2 = \text{const} \ll v_\parallel^2$. Hence, the electron motion is practically one dimensional. The electron drift velocity is very close to the total electron velocity and the ionization rate is small.

In the case where $N\sigma_{mt}\varepsilon/eE \gg 1$ the distribution function is almost isotropic and we can use the two spherical harmonic expansion: $f_e = f_0 + f_1 \cos\theta$. In this case the drift velocity is much smaller than the total velocity, and the particle spends more time at each position, which means that the ionization rate is high.

It is easy to show (see Fig. 3) that if eE/N even slightly exceeds the max $[\varepsilon\sigma_{mt}(\varepsilon)]$ at the cathode, then it exceeds it almost up to the point where $E = 0$. This means that electrons from the cathode form a beam which moves toward the anode almost without losses up to the end of the cathode fall, where the electric field is essentially zero. In the region, where eE/N is less than $\varepsilon\sigma_{mt}(\varepsilon)$ they begin to lose their speed and, therefore, produce more ions.

Let us now find the energy distribution of electrons in the high field region. Taking into account the one-dimensional character of electron motion we obtain for the distribution function $f_e(\mathscr{E}, \theta = 0; x)$ in the shell $(\mathscr{E}, \mathscr{E} + d\mathscr{E})$:

$$\frac{\partial f_e}{\partial x} = N\sigma_{il}(f'_e - f_e) + R_i(x) m_e \delta(\varepsilon). \qquad (15)$$

Here $R_i(x)$ is the ionization term:

$$R_i(x) = (N/m_e) \int_I^{eU-V} f_e(\varepsilon)\sigma_i(\varepsilon) \, d\varepsilon.$$

For the shell which has the maximum energy ($\mathscr{E} = eU$), $\varepsilon = \varepsilon_{max} = eU - V$) we immediately find

$$f_0(x) = f_0(x_0) \exp[-N\sigma_{il}(x - x_0)], \qquad (16)$$

where $x_0 = I/|eE(0)|$. Since $x_0 \ll (N\sigma_{il})^{-1}$ we can neglect x_0 in the exponent, and have

$$f_0(x_0) = \frac{\gamma}{1+\gamma} \frac{j\delta(\varepsilon - I)}{\sqrt{2I/m}\sqrt{1 - (N\sigma_{mt} I)/|eE(0)|}} \approx \frac{\gamma}{1+\gamma} \frac{j\delta(\varepsilon - I)}{\sqrt{2I/m}}. \qquad (17)$$

Here we used the boundary condition at the cathode: $j_e(0) = \gamma j_i(0)$ and the condition for constant electron current in this region, $j_e = \gamma j/(1+\gamma) = \text{const}$. For the second peak (with the energy $\varepsilon_{max} - I$) we can neglect the $R_i(x)$ term compared to the term $N\sigma_{il} f(\varepsilon_{max}; x)$. Thus,

$$f_e(\varepsilon_{max} - I, x) \approx (x - x_0) N\sigma_{il} f_0(x) = (x - x_0) N\sigma_{il} \exp(-N\sigma_{il}(x - x_0)) f_0(x_0). \qquad (18)$$

For the wide part of the distribution function which is below ε_{max}, down about 50 eV or more, we can expand the

* This can be shown by direct integration of the analytic expression [2] for elastic cross-section $\sigma(\varepsilon, \theta) = \exp[-a\varepsilon^{1/2}\sin(\theta/2)]$ given as an approximation to the experimental results [7]. There is, however, some discrepancy between this result and the data given by Hayashi [8], where $\sigma_{mt}(\varepsilon) \propto \varepsilon^{-3/2}$.

difference

$$f'_e - f_e \approx I \frac{\partial f_e}{\partial \varepsilon}. \tag{19}$$

After substituting (19) to (15) and using $x = l(1 - \sqrt{V/eU})$, we find that $f_e(\mathscr{E}, x)$ must be a function of the combination $\mathscr{E} - I(l/\lambda_{il})\sqrt{V/eU}$:

$$f_e(\mathscr{E}, x) = f_e\left(\mathscr{E} - I \frac{l}{\lambda_{il}} \sqrt{\frac{V}{eU}}\right). \tag{20}$$

The value of this function for any particular value of the argument can be determined at the point on the trajectory where the particles were produced. Integrating eq. (15) over the small distance Δx near the point, where the particle is created ($v = 0$) and using simple relations between δ-functions: $\delta(\varepsilon) = \delta(\mathscr{E} - V) = \delta(x - x^*)/(eE^*)$ we find

$$f(\varepsilon = +0; x) \equiv f(\mathscr{E}, x^*) = \frac{R_i(x^*) m_e}{eE^*}. \tag{21}$$

The asterisk means that a function is evaluated at the point $\varepsilon = 0$: $x^* = x(V = \mathscr{E})$, $E^* = E(x^*) = E(V = \mathscr{E})$.

4. Ionization and excitation rates

For the parameters of the Den Hartog *et al.* experiments [1]: $N \sim 10^{17}\,\mathrm{cm}^{-3}$, $\sigma_{il} \sim (3\text{–}5) \cdot 10^{-17}\,\mathrm{cm}^2$, $l \approx 0.3\,\mathrm{cm}$, we have $l/\lambda_{il} \sim 1$. This means that particles in the tail do not lose much of their energy in the CF region ($\Delta\mathscr{E} \leqslant I \ll \mathscr{E}$), and as long as electric field is strong enough the particles do not scatter. Each energetic particle produces the same number of the new ones per unit length ($1/\lambda_i = N\sigma_i$) independently of its own energy. The electron current j_e, ionization R_i and excitation R_{ex} rates depend on the coordinate x as follows,

$$j_e(x) = j_e(0) \exp\left[(x - x_0)/\lambda_i\right] \approx j_e(0) \exp(x/\lambda_i), \tag{22}$$

$$R_i(x) = N\sigma_i\, j_e(\varepsilon > I)/e \approx N\sigma_i\, j_e/e \approx N\sigma_i \frac{j_e(0)}{e} \exp(x/\lambda_i), \tag{23}$$

$$R_{ex}(x) = N\sigma_{ex}\, j_e(\varepsilon > I)/e$$

$$\approx N\sigma_{ex}\, j_e/e \approx N\sigma_{ex} \frac{j_e(0)}{e} \exp(x/\lambda_i). \tag{24}$$

Indeed, let us divide current $j_e(x)$ into two parts: $j_-(x)$ and $j_+(x)$ produced by the particles with lower and higher kinetic energies than I, respectively. For these currents we can write

$$\frac{\partial j_+}{\partial x} = \frac{eE}{I} j_-, \tag{25}$$

$$\frac{\partial j_-}{\partial x} = N\sigma_i j_+ - \frac{eE}{I} j_-. \tag{26}$$

Since $eE \gg IN\sigma_i$ we immediately obtain expression (22). If we remember also that in the quasi one-dimensional case

$$R_{i,ex} = N \int_I^\infty f_e v \sigma_{i,ex}\, dv = N\sigma_{i,ex} j_+/e, \tag{27}$$

and that $j_- \ll j_+$, then we obtain (23) and (24). Having found an expression for the ionization rate and general form of solution (20), we can easily find the electron distribution function at any point x as a function of kinetic energy ε. Using (20) and (21), and neglecting the changes in the particle energy along its trajectory, we can write:

$$f(\varepsilon; x) \equiv f[\mathscr{E}, x^*(x)] \frac{dx^*}{dx} = \frac{R_i(x^*)}{eE^*} \frac{dx^*}{dx}$$

$$= \frac{R_i(x)}{eE} \frac{R_i(x^*)}{R_i(x)} \frac{E}{E^*} \frac{dx^*}{dx}$$

$$= f(\varepsilon = 0, x) \frac{V}{V + \varepsilon} \exp(-\Delta x/\lambda_i). \tag{28}$$

In the above we used $V^* = V + \varepsilon$;

$$\Delta x = l(\sqrt{(\varepsilon + V)/eU} - \sqrt{V/eU}). \tag{29}$$

Note, that in the vicinity of the zero-field point condition (9) is not satisfied. However, the particles which already gained energy still move forward almost one-dimensionally, and slowly increase the angular width of their distribution function.

At the end of CF ($E = 0$) the electron distribution function is [$V = 0$ in (28)]

$$f(\varepsilon) = \frac{\text{const.}}{\varepsilon} \exp\left(-\frac{l}{\lambda_i}\sqrt{\frac{\varepsilon}{eU}}\right). \tag{30}$$

Combining eqs (19) and (15) with the condition $E = 0$ we obtain equation

$$\frac{\partial f}{\partial x} - IN\sigma_{il} \frac{\partial f}{\partial \varepsilon} = 0, \tag{31}$$

which gives us characteristics $\varepsilon + Ix/\lambda_{il} = \text{constant}$ and solution

$$f(\varepsilon, x) = f\left(\varepsilon + I \frac{x - l}{\lambda_{il}}\right), \tag{32}$$

with boundary condition that at $x = l$ the functions (32) and (30) are the same. The coordinate dependence of ionization and excitation rates here are

$$R_{i,ex} \propto \int_{I(1 + x/\lambda_{il})}^{U} \exp\left(-\frac{l}{\lambda_i}\sqrt{\frac{\varepsilon}{eU}}\right) \frac{\sigma_{i,ex}}{\varepsilon}\, d\varepsilon. \tag{33}$$

Formulae (23), (24) and (33) qualitatively well describe the distributions of excitation and ionization rates in a low pressure discharge [1].

5. Conclusions

We have presented a simple kinetic model for a high electric field, or low pressure DC discharge. Simplified, but realistic assumptions lead to analytic expressions for the electric field, the electron distribution function and ionization and excitation rates. The calculated electric field was compared with some experiments done by the Wisconsin group [1]. Reasonable agreement was found. In addition, the analytic expressions for the rates suitably describe the experiments. The case for moderate electric fields is under study.

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